Fractal modeling of the J–R curve and the influence of the rugged crack growth on the stable elastic–plastic fracture mechanics

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A B S T R A C T

In this paper, fractal geometry is used to modify the Griffith–Irwin–Orowan classical energy balance. Crack fractal geometry is introduced in the elastic–plastic fracture mechanics by means of the Eshelby–Rice J-integral and the influence of the ruggedness of the crack surface on the quasistatic crack growth is evaluated. It is shown that the rising of the J–R curve correlates to the topological ruggedness dimension of the crack surface. Results from fracture experiments are shown to be very well fitted with the proposed model, which is shown to be a unifying approach for fractal models currently used in fracture mechanics.

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1. Introduction

Classical elastic–plastic fracture mechanics (EPFM) quantifies velocity and energy dissipation of a crack growth in terms of the projected lengths and areas along the growth direction. However, in the fracture phenomenon, as in nature, geometrical forms are normally irregular and not easily characterized with regular forms of Euclidean geometry. As an example of this limitation, there is the problem of stable crack growth, characterized by the J–R curve [1]. The rising of this curve has been analyzed by qualitative arguments [1–4] but no definite explanation in the realm of EPFM has been provided.

Alternatively, fractal geometry can reveal aspects that traditional Euclidean geometry cannot [5], and knowing how to calculate the true lengths, areas and volumes of irregular elements, such as fractured surfaces and others, is very important for the purpose suggested in this work. The different geometric details contained in the fracture surface tell the history of the crack growth and the difficulties encountered during the fracture process [6]. For this reason many scientists have worked on the fractal characterization of the fracture surface (fractography) [7–13]. Also, it became necessary to include the topology of the fracture surface into the classical fracture mechanics theory [5,9–17]. This new “Fractal Fracture Mechanics” follows the fundamental basis of the classical fracture mechanics, changing some of its equations by taking into consideration the fractal aspects of the fracture surface with analytical expressions [18].

There have been several proposals for including the fractal theory into de fracture mechanics in the last three decades. Among these is the proposal by Williford [14], which establishes a relationship between fractal geometric parameters...
Nomenclature

A1 and A2 group of welds
Apl plastic area defined by the plot of stress versus displacement
BN net thickness of the testing sample
C boundary curve
D fractal dimension
DCT1, DCT2 group of samples
dV infinitesimal volume encapsulated by C
dLo incremental growth of the crack length
E Young’s elastic modulus
F the work performed by external forces
f, g general functions
Go elastic energy release rate
H Hurst’s exponent
\(\Delta H_0\) vertical plane projected crack length J-Eshelby–Rice J-integral on the stretched rugged crack path
Jo elastic and plastic energy release rate, Eshelby–Rice J-integral on the plane projected crack path
JC critical value of the J–R curve
JoC critical value of the J–R curve for the Mode – I fracture
Kc elastic stress intensity factor
Kr crack resistance on the rugged crack path
Kco critical value of the crack resistance on the rugged crack path
Kco fracture toughness
L rugged crack length
Lo horizontal plane projected crack length
Loc Griffith’s critical crack length
\(\Delta L\) rugged crack length
\(\Delta L_o\) distance between two points of the crack (the projected length of the crack)
LoC critical crack length
lo fractal cell size or minimal crack length
Nh = \(\Delta L_o/lo\) number of units of the crack length in the growth direction
\(N_v = \Delta H_0/ho\) number of units of the crack length in the perpendicular direction which crack grows
index “o” denote measurements taken on the projected plane
PU1, PU2 group of polymer samples
Ro crack resistance, per unit thickness
s distance along the boundary
T tensile stress
u displacement of the application point of forces
ULo change in the elastic strain energy caused by the introduction of a crack with length \(\Delta L_o\) in the sample
ULo elastic contribution to the strain energy in the material
UV strain energy density integral of the rugged crack
UVo strain energy density integral of the plane projected crack
Uj energy to create two new fracture surfaces, given by the product of the specific elastic surface energy of the material, \(\gamma_e\), by the surface area of the crack (two surfaces of length \(\Delta L_o\))
Up pl plastic contribution to the strain energy in the material
V volume encapsulated by C
x horizontal fixed x-coordinate
y vertical fixed y-coordinate
x' mobile x-coordinate on the rugged crack path
y' mobile y-coordinate on the rugged crack path
Yo form function defined by the shape of the sample
w width of the testing sample
W strain energy density by volume

Greek letters
\(\varepsilon\) scale \((lo/Lo)\) of the fractal scaling
\(\varepsilon_h, \varepsilon_v\) horizontal and vertical scale \((lo/Lo)\) of the fractal scaling
\(\varepsilon_iq\) strain field around the crack tip
\(\gamma_e\) specific elastic surface energy of the material
\(\gamma_p\) specific plastic deformation surface energy
and parameters measured in fatigue tests. Using Williford’s proposal, Gong and Lai [15] developed one of the first mathematical relationships between the fracture resistance \( J – R \) curve and the fractal geometric parameters of the fracture surface. Later, Mosolov and Borodich [17] established mathematical relationships between the elastic stress field around the crack and the rugged parameter of the fracture surface. Following a similar idea, Carpinteri and Chiaia [19] described the fracture resistance behavior as a consequence of its self-similar fractal topology. They used Griffith’s theory and found a relationship between the \( J \)-curve and the advancing length of the crack and the fractal dimension. Despite the non-differentiability of the fractal functions, they were able to obtain this relationship through a renormalizing method. Other formulations have been proposed by Bouchaud [20] using the correlation between ruggedness heights at different coordinates in the fracture surface. Recently, Alves [21,22] presented a self-affine fractal model, capable of describing fundamental geometric properties of the fracture surface, including the local ruggedness in Griffith’s criterion. In all these formulations fractal theory was introduced in an analytical context in order to establish a mathematical expression for the fracture resistance curve, putting in evidence the influence of the crack ruggedness.

The objective in this work is to introduce fractal theory into the elastic and plastic energy release rates, \( G_o \) and \( J_o \), in a novel way compared to other authors [9–16,23]. The non-differentiability of the fractal functions is avoided by developing a differentiable analytic function for the rugged crack length, which was previously applied to the \( G_o \) curve in the LEFM [21]. The proposed procedure changes the classical expression for \( G_o \), which is linear with the length of the fracture, into a non-linear equation. Also, the same approach is extended and applied to the Eshelby–Rice non-linear \( J \)-integral [24]. The new equations reproduce accurately the growth process of cracks in brittle and ductile materials. Through algebraic manipulations, it was possible to separate the energetics of the geometric part of the fracture process in the \( J \)-integral, thus explaining the registered history of the elastic and plastic strains left on the fracture surfaces during the fracture phenomenon. Also, the micro and macroscopic parts of the \( J \)-integral are distinguished. A generalization for the fracture resistance \( J – R \) curve for different materials is presented, dependent only on the material properties and the rugged geometry of the fractured surface.

Finally, it is demonstrated how the proposed model can contribute to a better understanding of certain aspects of the standard ASTM test [18].

2. Development of the fractal theory for the fracture phenomenon

2.1. The self-affine fractal model of a fracture surface

A fracture surface is considered a self-affine fractal object in different scales [5,7,25,26] when every region has the same statistical geometric characteristics as any other. Thus, a fractal model has been proposed [21,27] to calculate the rugged crack length \( L \) based on the Voss [28] equation, where the crack is compared to a self-affine noise in time, as shown by the following expression:

\[
\Delta L = \Delta L_o \sqrt{1 + \left( \frac{\Delta H_o}{l_o} \right)^2 \left( \frac{l_o}{\Delta L_o} \right)^{2H-2}}
\]

and the scheme in Fig. 1, where \( \Delta L \) is the increment of the rugged length, \( \Delta L_o \) the horizontal projection of the crack length extension (excluding the initial crack), \( \Delta H_o \) the vertical projection of the crack length increment, \( l_o \) the unit length which defines the scale \( \varepsilon = l_o/\Delta L_o \) under which the crack profile is scrutinized, and \( H \) is Hurst’s exponent. As a general rule the index “\( o \)” is adopted to denote measurements taken on the projected plane (see Figs. 1 and 2).

Many fracture surfaces grow in a vertical direction, maintaining a constant horizontal length [29] and the fracture surface can be understood as spreading in the horizontal direction, while its ruggedness is developed in the vertical direction, as shown in Fig. 1a. A simplified mathematical expression for \( \Delta L \) can be obtained taking rectangular boxes to count fractal cells with the size \( l_o \), where \( \Delta H_o = l_o \), resulting in

\[
\Delta L = \Delta L_o \sqrt{1 + \left( \frac{l_o}{\Delta L_o} \right)^{2H-2}}
\]

Hurst’s exponent \( H \) is related to the fractal dimension \( D \) through \( H = 2 – D \).
Eq. (2) can be used in the calculation of the local ruggedness \( n \) of the crack.

\[
\frac{dL}{dL_0} \equiv \lim_{l_0 \to 0} \frac{\Delta L_{l_0+l_0} - \Delta L_{l_0}}{l_0}
\]

and the local ruggedness for a fracture surface can be written as

\[
\xi \equiv \frac{1 + (2 - H)\left(\frac{L_0}{\Delta L_0}\right)^{2H-2}}{1 + \left(\frac{L_0}{\Delta L_0}\right)^{2H-2}} \geq 1.
\]

Observe that it is necessary to choose a geometric method capable of measuring the increasing \( dL \) in time (or load) starting from a fixed origin and variable increments \( AL_0 \). Eq. (4) is suggested to modify the classical fracture mechanics, introducing fractal geometry in its equations to portray the crack growth as a fractal growth, as described in the next section.

To describe the crack growth along with the fractal growth measurements, the Sand-Box or the Box-Counting methods can be used.

Supposing that the crack extension \( AL_0 \) is considered increasing continuously in time, \( AL_0 = AL_0(t) \), with incremental stretches \( l_0 \), the fracture is formed continuously in a similar way to the continuous appearance of self-affine fractal seeds, with horizontal projected size \( L_0 \) in a fractal growth (Fig. 1a). Therefore, the counting of the number of seeds with the size \( AL_0 = l_0 \) added on the fractal structure of the crack surface, when it propagates, can be done using the Sand-Box-counting method. In this method, the counting is done by fixing one edge of a scaling box at the origin \( O \) of the crack, and extending the opposite edge by \( AL_0 \) on the crack tip, simultaneously accompanying the advance of the crack as it is shown in Fig. 1a. In this way, the rugged and the projected lengths, \( AL \) and \( AL_0 \), respectively, are obtained as a function of time, i.e. \( AL = AL(t) \) and \( AL_0 = AL_0(t) \).
The Sand-Box method is the best method since it adapts to the condition of the continuous crack growth as it begins from a minimum fixed crack length \( L_0 \). This minimal crack length can be understood as being the unitary increment, \( AL_o = L_0 \), of the function \( AL = f(AL_o) \), while \( AL_o \) is variable, allowing the computation of the derivative according to Eq. (3). On the other hand, the Box-Counting method doesn’t possess this resource since it already begins from a formed crack of fixed length \( AL_o \) which is subdivided into boxes of variable size \( b \).

2.2. The \( J_o \)-Eshelby–Rice integral for rugged and plane projected crack paths

If the fracture surface is characterized by a crack with length \( AL_o \) and a unit thickness body, the \( J_o \)-integral, or the path-independent integral for the plane projected crack path was introduced by Rice in 1968 [24].

\[
J_o = -\frac{d\Pi_o}{dL_o} = -\left( \int_V W \frac{dx}{dL_o} dy - \int_C \hat{T} \cdot \frac{\partial u}{\partial L_o} ds \right) \tag{5}
\]

where \( \Pi_o \) is the potential energy in the volume \( V_o \) encapsulated by the boundary \( C \) enclosing the crack tip, \( dL_o \) the incremental growth of the crack length, \( W \) the internal work and \( T \) and \( u \) are the tractions and displacements along the boundary \( C \). For a fixed boundary \( C \), \( d/dL_o = -d/dx \) and the \( J_o \)-integral can be written only in terms of the boundary:

\[
J_o = \int_C Wdy - \int_C \hat{T} \cdot \frac{\partial u}{\partial x} ds \tag{6}
\]

Now, the \( J-R \) Eshelby–Rice integral theory is modified to include fracture surface ruggedness based on the fractal considerations of fracture mechanics. Therefore, to describe the energy dissipation process in a rugged crack path it is necessary to postulate that:

(I) The energy to create the fracture surface on the rugged path is the same on the projected crack path, \( L_o \), then, the volumetric strain energy \( U_{vol} = U_r \) and the potential energy \( \Pi_o = \Pi \) (energy equivalence previously proposed by Irwin);

(II) Postulate that fracture mechanics mathematical formalism is invariant on a rugged crack path and on a projected crack path.

Rewriting the terms

\[
\frac{dx}{dL_o} = \frac{dx}{dt} \frac{dt}{dL_o}; \quad \frac{\partial u}{\partial L_o} = \frac{\partial u}{\partial x} \frac{dx}{dL_o}
\]

and substituting in Eq. (5), on the same volume \( V_o = V \) and boundary \( C \) (see Fig. 2), one can write

\[
J_o = -\left( \int_V W \frac{dx}{dL_o} dy - \int_C \hat{T} \cdot \frac{\partial u}{\partial x} ds \right)
\]

From the second postulate, the new \( J \)-integral on the rugged crack path is given by:

\[
J = -\frac{d\Pi}{dL} = -\frac{d}{d\xi} \left( \int_V Wd\xi dy - \int_C \hat{T} \cdot \frac{\partial u}{\partial x} ds \right)
\]

or

\[
J = -\frac{d\Pi}{dL} = -\left( \int_V W \frac{d\xi}{dL} dy - \int_C \hat{T} \cdot \frac{\partial u}{\partial \xi} ds \right)
\]

where the * symbol represents coordinates with respect to the rugged path. Therefore, in an analogous way to the \( J \)-integral for the projected crack path given by Eq. (6), since \( d/dL = -d/d\xi \), one has

\[
J = \int_V Wd\xi - \int_C \hat{T} \cdot \frac{\partial u}{\partial \xi} ds
\]

Returning to Eq. (6) considering the first postulate along with the derivative chain rule and substituting in Eq. (10), one has

\[
J_o = -\frac{d\Pi}{dL_o} = -\left( \int_V W \frac{d\xi}{dL_o} dy - \int_C \hat{T} \cdot \frac{\partial u}{\partial \xi} ds \right) \frac{dL_o}{dL_o}
\]

Comparing Eq. (8) with Eq. (12) and considering that the rugged crack is a result of a transformation in the volume of the crack, analogous to “baker’s transformation” of the projected crack over the Euclidian plane, it can be concluded that

\[
\frac{d\xi}{dL_o} dy = \frac{\partial (\xi', \eta')}{\partial (\xi, \eta)} dx dy
\]

\[
\frac{d\xi}{dL_o} = \frac{dx}{dL_o} dy
\]

This minimal crack length can be understood as being the unitary increment, \( AL_o = L_0 \), of the function \( AL = f(AL_o) \), while \( AL_o \) is variable, allowing the computation of the derivative according to Eq. (3). On the other hand, the Box-Counting method doesn’t possess this resource since it already begins from a formed crack of fixed length \( AL_o \) which is subdivided into boxes of variable size \( b \).
which shows the equivalence between the volume elements,
\[ dV = dx' dy' = dx dy \] (14)
Therefore, the ruggedness \( dl/dlo \) of the rugged crack path does not depend on the volume \( V \), the boundary \( C \), nor on the infinitesimal element length \( ds \) or \( dy \). Thus, it must depend only on the characteristics of the rugged path described by the crack on the material. Finally, the integral in Eq. (12) can be written as
\[ J_0 = \left( \int V W \frac{dx}{dl} dy - \int C \bar{t} \cdot \frac{\partial \bar{u}}{\partial \bar{e}} ds \right) \frac{dl}{dl_o} \] (15)
or alternatively,
\[ J_0 = \left( \int V W dy' - \int C \bar{t} \cdot \frac{\partial \bar{u}}{\partial \bar{e}} ds \right) \frac{dl}{dl_o} \] (16)
where the infinitesimal increments \( dx/dl = -\cos \theta_i \) and \( dy = -dy \cos \theta_i \) accompany the direction of the rugged path \( L \) as show in Fig. 2. Therefore,
\[ J = \int V W dy \cos \theta_i - \int C \bar{t} \cdot \frac{\partial \bar{u}}{\partial \bar{e}} ds \] (17)
Observe that the \( J \)-integral for the rugged crack path given by Eq. (17) differs from the \( J \)-integral for the plane projected crack path given by Eq. (6), just by a fluctuating term \( \cos \theta_i \) inside the integral. Also, as can be observed from Eqs. (6) and (12) the influence of the ruggedness of the material in the elastic–plastic energy release rate is
\[ J_0 = \int \frac{dl}{dl_o} \] (18)
It must be pointed out that this relationship is general and the introduction of the fractal approach to describe the ruggedness \( dl/dlo \) is just one particular way of modeling.

2.3. Influence of ruggedness in mathematical relationships of fracture mechanics

In practice, \( G–R \) or \( J–R \) curves are obtained experimentally by plotting the energy release rates \( G_o \) or \( J_o \) and \( R_o \) against the projected crack length \( L_o \). Instability occurs at \( G_o = R_o = J_o = dR_o/dL_o \) or \( dl_o/dL_o \) and some modifications are introduced in the classical fracture mechanics equations. Based on Griffith–Irwin–Orowan’s energy balance [1] for stable crack growth [3,30], the crack resistance \( R_o \) per unit thickness is adjusted to
\[ R_o = \frac{dU}{dl_o} \frac{dl}{dl_o} = (2\gamma_e + \gamma_p) \frac{dl}{dl_o} \] (19)
where \( U \) is the product of the specific elastic surface energy \( \gamma_e \) of the material by the surface area of the crack (two surfaces of length \( \Delta L_o \)) plus the contribution of the specific plastic deformation energy \( \gamma_p \). The elastic–plastic energy release rate \( J_o \) is adjusted in the same way
\[ J_o = \frac{d(F - U)}{dl} \frac{dl}{dl_o} \] (20)
The energy balance proposed by Griffith–Irwin–Orowan, for stable fracture is
\[ J_o = R_o \] (21)
Therefore, for plane stress or plane strain conditions, one can write,
\[ J_{Ro} = (2\gamma_e + \gamma_p) \frac{dl}{dl_o} = \frac{K_{Ro}^2 f(v)}{E} \] (22)
where \( f(v) \) is a function that defines the testing condition. For plane strain, \( f(v) = 1 \), and for plane stress, \( f(v) = 1 - v^2 \). Thus,
\[ K_{Ro} = \sqrt{\frac{(2\gamma_e + \gamma_p)E}{f(v)}} \frac{dl}{dl_o} \] (23)
Knowing that fracture toughness is given by
From classical fracture mechanics, the fracture resistance for loading mode I is given by

\[ K_{R_a} = Y_o \left( \frac{L_o}{W} \right) \sigma_f \sqrt{L_o} \]  

(26)

where \( Y_o \left( \frac{L_o}{W} \right) \) is a function that defines the shape of the sample (CT, SEBN, etc.) and the type of test (traction, flexion, etc.) and \( \sigma_f \) is the fracture stress. Considering the case when \( L_o = L_{oc} \), then \( K_{Rao} = K_{ICO} \) and the fracture toughness for loading mode I is given by

\[ K_{ICO} = Y_o \left( \frac{L_{oc}}{W} \right) \sigma_f \sqrt{L_{oc}} \]  

(27)

Therefore, from Eq. (23) the fracture toughness curve for loading mode I is given by

\[ K_{Rao} = K_{ICO} \left( \frac{dL}{dL_o} \right) \]  

(28)

Substituting Eq. (26) in Eq. (23), one has

\[ \sqrt{\frac{(2\gamma_{ce} + \gamma_p)E}{f(v)}} \frac{dL}{dL_o} = Y_o \left( \frac{L_o}{W} \right) \sigma_f \sqrt{L_o} \]  

(29)

or

\[ \frac{dL}{dL_o} = Y_o^2 \left( \frac{L_o}{W} \right) \frac{\sigma_f^2 L_o f(v)}{(2\gamma_{ce} + \gamma_p)E} \]  

(30)

Observe that according to the right-hand side of Eq. (30), the ruggedness \( \frac{dL}{dL_o} \) is determined by the condition of the test (plane strain or stress), the shape of the sample (CT, SEBN, etc.) the type of test (traction, flexion, etc.) and the kind of material.

Considering the fracture surface as a fractal topology, such as in the previously developed models [20,21,27,31] for the left-hand side of Eq. (30), one observes that the characteristics of the fracture surface listed above are all included in the ruggedness fractal exponent \( H \). Therefore, substituting Eq. (2) in Eq. (22), one obtains

\[ J_{Rao} = (2\gamma_{ce} + \gamma_p) \left( 1 + (2 - H) \left( \frac{\beta}{A_{cr}} \right)^{2H-2} \right)^{2H-2} \]  

(31)

Eq. (31) is non-linear in the crack extension \( \Delta L_{cr} \). It corresponds to the classical Eq. (21) corrected for a rugged surface with Hurst’s exponent \( H \). Its graph is shown in Fig. 3.

![Fig. 3. Graph of the curve J–R obtained by the model fractal of the fracture surface for different Hurst rugosity exponents.](image-url)
The $J$-integral on the rugged crack path is a specific characteristic of the material. Therefore, it can be considered as being proportional to $J_C$ [18] on the onset of crack extension, since that in this case it has $L \gg L_o$. From Eq. (18), one has

$$J_o \sim J_C \frac{dl}{dl_o}$$  \hspace{1cm} (32)

Substituting the monofractal crack model proposed in Eq. (2), for the ruggedness $dl/dl_o$, one has

$$J_o \sim J_C \frac{1 + (2 - H) \left( \frac{l_o}{l} \right)^{2H - 2}}{\sqrt{1 + \left( \frac{l_o}{l} \right)^{2H - 2}}},$$  \hspace{1cm} (33)

corroborating the fact that the surface specific energy is related to the critical fracture resistance

$$J_C \sim (2 \gamma_c^e + \gamma_p)$$  \hspace{1cm} (34)

Case-1.

The local self-similar limit can be calculated applying the condition when $H_o \sim l_o$ in Eq. (1), obtaining

$$J_{Ro} = (2 \gamma_c^e + \gamma_p) (2 - H) \left( \frac{l_o}{l} \right)^{H - 1}$$  \hspace{1cm} (35)

This result corresponds to the same equation found by Mu and Lung [32] and Lung and Mu [33] and other authors for ductile materials.

Case-2.

The global self-affine limit of $J_o$ can be calculated by applying the condition when $H_o \rightarrow l_o \ll l_o$ in Eq. (1), obtaining the linear elastic result, where $J_o = G_o$ and

$$G_{Ro} = 2 \gamma_c^e + \gamma_p$$  \hspace{1cm} (36)

can be understood as the general case presented for brittle materials such as glass and ceramics.

3. Experimental procedure

With the purpose of applying the proposed fractal model an experimental procedure for obtaining the experimental $J$–$R$ curves is presented.

3.1. The samples

The samples used in this work are multipass High Strength Low Alloy (HSLA) steel weld metals, which are divided in two groups based on the welding process utilized and the microstructural composition. The first group (A1 and A2 welds) is composed of C-Mn Ti-Killed weld metals and were joined by a manual metal arc process. The second group (B1 and B2 welds), joined by a submerged arc welding process, is also a C-Mn Ti-Killed weld metal, but with different alloying elements added to increase the ability to harden. The chemical composition and tensile properties of both weld metals are listed in Tables 1 and 2, respectively. This table is used in the calculations of $J_C$ and other physical quantities requested by classical fracture mechanics to fit the equations yielded by the proposed model with the experimental results.

<table>
<thead>
<tr>
<th>Weld</th>
<th>C (wt.%)</th>
<th>Mn</th>
<th>Ti</th>
<th>Ni</th>
<th>Cr</th>
<th>Al</th>
<th>Cu</th>
<th>N</th>
<th>B</th>
<th>Si</th>
<th>Mo</th>
<th>V</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.07</td>
<td>1.4</td>
<td>300</td>
<td>0.85</td>
<td>–</td>
<td>80</td>
<td>350</td>
<td>70</td>
<td>&lt;5</td>
<td>0.42</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>A2</td>
<td>0.07</td>
<td>1.4</td>
<td>230</td>
<td>0.83</td>
<td>–</td>
<td>70</td>
<td>400</td>
<td>75</td>
<td>&lt;5</td>
<td>0.42</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>B1</td>
<td>0.05</td>
<td>1.07</td>
<td>490</td>
<td>2.38</td>
<td>0.60</td>
<td>110</td>
<td>650</td>
<td>55</td>
<td>–</td>
<td>0.58</td>
<td>0.55</td>
<td>230</td>
<td>380</td>
</tr>
<tr>
<td>B2</td>
<td>0.05</td>
<td>1.32</td>
<td>760</td>
<td>2.65</td>
<td>0.05</td>
<td>100</td>
<td>750</td>
<td>120</td>
<td>–</td>
<td>0.85</td>
<td>–</td>
<td>–</td>
<td>230</td>
</tr>
</tbody>
</table>

Obs.: The amount of S and P were in the range of 0.22–0.024 wt.%.  
* The values are in ppm.

<table>
<thead>
<tr>
<th>Weld</th>
<th>A1</th>
<th>A2</th>
<th>B1</th>
<th>B2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield stress (MPa)</td>
<td>516</td>
<td>484</td>
<td>771</td>
<td>757</td>
</tr>
<tr>
<td>Tensile strength (MPa)</td>
<td>771</td>
<td>577</td>
<td>909</td>
<td>798</td>
</tr>
</tbody>
</table>
3.2. The J–R curve testing

The fracture toughness evaluation was executed using the $J$-integral concept and the elastic compliance technique with partial unloadings of 15% of the maximum load. The tests were executed in a Material Test System (MTS810) system at an ambient temperature, according to standard ASTM E1737-96 [18]. A single edge notch bending, SENB, and compact tension, CT, were used and both specimens contained side grooves and had 7.5 and 18 mm of thickness, respectively. One $J$–$R$ curve for each tested specimen was retrieved. The fracture surface analysis was executed using scanning electronic microscopy, SEM.

4. Results

4.1. J–R curve tests and fitting

In Figs. 5 and 6, samples of the results of the tests for obtaining the $J$–$R$ curve of the metallic weld materials are shown. The values of $J_{IC}$ and $Loc$ were calculated in agreement with ASTM – 813-89 [34] are shown in the Table 3. Typically, in these figures, the $J$–$R$ curves measured experimentally were calculated using Eqs. (31) and (35), where the multiplying factor $2c/e + c/p$ was obtained together with the $lo$ and $H$ values for the different samples. These values were compatible with the experimental values obtained for ductile [18,34] and brittle materials [10,23]. The rising of the $J$–$R$ curves is due to Hurst’s exponent and it is accentuated as this exponent decreases from 1.0 to 0.0, as can be seen in Eq. (31) and in Fig. 3.

The columns of the Tables 4 and 5 were calculated in the following way: the experimental $J$–$R$ curves were fitted using Eqs. (31) and (35), determining the values of $2c_{eff}$, $H$ and $lo$. Making $J = 2c_{eff}$ in Eq. (35), the value of the crack size $Lo_{c_{eff}}$ was calculated in agreement with ASTM – 813-89 [34].

Table 3
Data from experimental testing of $J$–$R$ curves obtained by compliance method.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$\sigma_f$</th>
<th>$J_c$ (exp)</th>
<th>$Loc$ (exp)</th>
<th>$H$ (exp)</th>
<th>$H$ (exp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1CT2</td>
<td>516.00</td>
<td>291.60</td>
<td>0.48256</td>
<td>0.71 ± 0.01</td>
<td></td>
</tr>
<tr>
<td>A2SEB2</td>
<td>537.00</td>
<td>174.67</td>
<td>0.36264</td>
<td>0.77 ± 0.01</td>
<td></td>
</tr>
<tr>
<td>B1CT6</td>
<td>771.00</td>
<td>40.61</td>
<td>0.22634</td>
<td>0.77 ± 0.02</td>
<td></td>
</tr>
<tr>
<td>B2CT2</td>
<td>757.00</td>
<td>99.22</td>
<td>0.26553</td>
<td>0.58 ± 0.05</td>
<td></td>
</tr>
<tr>
<td>DCT1</td>
<td>554.00</td>
<td>227.00</td>
<td>0.40487</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>DCT2</td>
<td>530.00</td>
<td>211.47</td>
<td>0.3995</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>DCT3</td>
<td>198.75</td>
<td>318.00</td>
<td>1.00000</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>PU0.5</td>
<td>40.70</td>
<td>8.10</td>
<td>0.29951</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>PU1.0</td>
<td>40.70</td>
<td>3.00</td>
<td>0.23685</td>
<td>–</td>
<td></td>
</tr>
</tbody>
</table>

Table 4
Data from fitting of $J$–$R$ curves for the self-similar model.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$2c_{eff}$</th>
<th>$H$</th>
<th>$lo$ (mm)</th>
<th>$L_{c_{eff}} = l_o (2 - H)^{1/(H-1)}$</th>
<th>$C_1/2c_{eff} = (2 - H)^{1/(H-1)}$</th>
<th>$J_{L_{c_{eff}}^{(H-1)}} = cte$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1CT2</td>
<td>283.247</td>
<td>0.417</td>
<td>1.00944</td>
<td>0.495079</td>
<td>1.57411</td>
<td>445.862579</td>
</tr>
<tr>
<td>A2SEB2</td>
<td>187.639</td>
<td>0.208</td>
<td>0.82912</td>
<td>0.396956</td>
<td>2.07868</td>
<td>390.042318</td>
</tr>
<tr>
<td>B1CT6</td>
<td>40.514</td>
<td>0.573</td>
<td>0.51758</td>
<td>0.225686</td>
<td>1.89071</td>
<td>76.600193</td>
</tr>
<tr>
<td>B2CT2</td>
<td>101.204</td>
<td>0.592</td>
<td>0.64484</td>
<td>0.278764</td>
<td>1.68407</td>
<td>170.437782</td>
</tr>
<tr>
<td>DCT1</td>
<td>230.843</td>
<td>0.426</td>
<td>0.91887</td>
<td>0.416893</td>
<td>1.65219</td>
<td>381.397057</td>
</tr>
<tr>
<td>DCT2</td>
<td>209.127</td>
<td>0.461</td>
<td>0.87082</td>
<td>0.391328</td>
<td>1.65806</td>
<td>346.745888</td>
</tr>
<tr>
<td>DCT3</td>
<td>317.819</td>
<td>0.393</td>
<td>2.18249</td>
<td>0.990962</td>
<td>1.00057</td>
<td>318.000000</td>
</tr>
<tr>
<td>PU0.5</td>
<td>17.4129</td>
<td>0.476</td>
<td>2.88612</td>
<td>1.291434</td>
<td>0.87464</td>
<td>15.230001</td>
</tr>
<tr>
<td>PU1.0</td>
<td>2.95252</td>
<td>0.503</td>
<td>0.51653</td>
<td>0.229374</td>
<td>2079</td>
<td>6.138287</td>
</tr>
</tbody>
</table>

Table 5
Data from fitting of $J$–$R$ curves for the self-affine model.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$2c_{eff}$</th>
<th>$H$</th>
<th>$lo$ (mm)</th>
<th>$L_{c_{eff}} = l_o (2 - H)^{1/(H-1)}$</th>
<th>$C_1/2c_{eff} = (2 - H)^{1/(H-1)}$</th>
<th>$J_{L_{c_{eff}}^{(H-1)}} = cte$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1CT2</td>
<td>160.640</td>
<td>0.609</td>
<td>0.24422</td>
<td>0.105004</td>
<td>2.413408</td>
<td>387.700806</td>
</tr>
<tr>
<td>A2SEB2</td>
<td>102.750</td>
<td>0.442</td>
<td>0.31002</td>
<td>0.140040</td>
<td>2.930902</td>
<td>307.535922</td>
</tr>
<tr>
<td>B1CT6</td>
<td>22.980</td>
<td>0.700</td>
<td>0.08123</td>
<td>0.033873</td>
<td>2.757772</td>
<td>61.385976</td>
</tr>
<tr>
<td>B2CT2</td>
<td>57.978</td>
<td>0.705</td>
<td>0.10304</td>
<td>0.042893</td>
<td>2.529433</td>
<td>146.651006</td>
</tr>
<tr>
<td>DCT1</td>
<td>129.850</td>
<td>0.599</td>
<td>0.23309</td>
<td>0.100540</td>
<td>2.511844</td>
<td>326.184445</td>
</tr>
<tr>
<td>DCT2</td>
<td>118.850</td>
<td>0.624</td>
<td>0.20167</td>
<td>0.086294</td>
<td>2.512302</td>
<td>298.592197</td>
</tr>
<tr>
<td>DCT3</td>
<td>178.810</td>
<td>0.612</td>
<td>0.5282</td>
<td>0.228901</td>
<td>1.778386</td>
<td>318.000000</td>
</tr>
<tr>
<td>PU0.5</td>
<td>750.0</td>
<td>0.604</td>
<td>0.56541</td>
<td>0.238775</td>
<td>1.619852</td>
<td>12.153707</td>
</tr>
<tr>
<td>PU1.0</td>
<td>1690</td>
<td>0.649</td>
<td>1.08987</td>
<td>0.064244</td>
<td>2.938220</td>
<td>4.971102</td>
</tr>
</tbody>
</table>
calculated and it corresponded to the value of that specific surface energy. Using the experimental value of $J_{IC}$, $L_{oc}$ and $H$, determined by the $J-R$ curve, the values of the constants in the last column of Tables 4 and 5 were calculated.

The values that best fit the curves are shown in Table 4 for the self-similar model of Eq. (35) and in Table 5 for the self-affine model of Eq. (31). The measured values of $H$ differ from the experimental measurements (see Tables 4 and 5) with errors of less than 20% for the first sample and 2% for the second one, approximately. The reason for this larger error for the first sample is due to the quality of its fractographic structure that does not present well defined "contrast islands" [21].

Observing from the results shown in Figs. 4–6 that the modification of the fracture mechanics by a fractal scaling law (self-affine or self-similar) suggests that the model of the $J-R$ curve is non-linear. This model reproduces the elastic–plastic results as well as the classical approach.

In the fractal model the necessary energy to grow a fracture is no longer proportional to the fracture area or to the length $L_o$, as it was in the case of the classical elastic linear fracture mechanics.

Even though the self-similar model fits as well as the self-affine model, as shown in the results in Figs. 5 and 6, there is some difference on the fitting, but it is almost imperceptible. This difference is smaller in the self-affine model and is due to the fact that the self-similar model introduces errors into the calculation, therefore underestimating the values of the specific surface energy $\gamma_{eff}$ and the minimum size of the microscopic fracture $L_o$, although it does not affect the value of the Hurst exponent $H$. It is important to remember that for a self-affine natural fractal such as a crack, the self-similar limit approach is only valid at the beginning of the crack growth process [35] and the self-affine limit is valid for the rest of the process. It is observed from the results shown before that the ductile fracture is closer to self-similarity while the brittle fracture is closer...
to self-affinity. This is because the box-sizes that must be taken in the fractal scaling of the crack are of the kind $H_o \to L_o \gg l_o$ in the ductile fracture, and $H_o \to l_o \gg L_o$ in the brittle fracture.

4.2. The analysis of the $J$–$R$ curves using the fractal model

Eq. (31) represents a self-affine fractal model and demonstrates that apart from the coefficient $H$, there is a certain “universality” or, more accurately, a certain “generality” in the $J$–$R$ curves. This equation can be rewritten using a factor of universal scale, $\varepsilon = l_o/L_o$, as

$$f(2\gamma_e + \gamma_p)J_o = \frac{J_o}{2(2\gamma_e + \gamma_p)} = \frac{1 + (2 - H)e^{2H-2}}{\sqrt{2(1 + e^{2H-2})}} = g(\varepsilon, H)$$

(37)

which is a valid function for every obtained experimental results shown in Fig. 7. This figure shows the existent relation between the energetic and geometric components of the fracture resistance of the materials, according to the Eq. (37). Therefore, the greater the material energy consumption in the fracture, straining it plastically, the longer will be its geometric path and, consequently, more rugged will be the crack.

Fig. 6. $J$–$R$ curve fitted in accordance with the self-similar and self-affine models presented in the Eqs. (17) and (21), respectively for the B2CT2 sample of the HSLA-Mn steel killed with titanium and others alloy elements to increase the temperability.

Fig. 7. Generalized graph of $J$–$R$ curves for different materials, modeled using the self-affine fractal geometry, in function of the scale factor, $\varepsilon$, of the crack length.
In the self-similar limit where \( l_o \ll L_o = H_o \), Eq. (35) is applicable and the energetic and geometric components are put in evidence in the equation below,

\[
J_o = \left(2 \gamma_{eo} + \gamma_p\right) (2 - H) \left(\frac{l_o}{L_o}\right)^{H-1}
\]

(38)

From Eq. (59), an expression can be derived which results in a constant value associated to each material,

\[
\frac{J_0 L_o^{H-1}}{\text{macroscopic}} = \frac{(2 \gamma_{eo} + \gamma_p)(2 - H) l_o^{H-1}}{\text{microscopic}} = (\text{const})_{\text{material}}
\]

(39)

It is possible to conclude that the macroscopic and microscopic terms on the left and right-hand sides of Eq. (39) are both equal to a constant, suggesting the existence of a fracture fractal property valid for the beginning of the crack growth, and justified experimentally and theoretically. These constant values were calculated for each point in each \( J-R \) curve for the tested materials. The average value for each material is listed in the last column of Tables 4 and 5. Observe that this new constant can be understood as a "fractal density of energy" and it is a physical quantity that takes into account the ruggedness of the fracture surface and other physical properties. Its existence can explain the reason for different properties encountered when defining the value of fracture toughness \( K_{IC} \). This constant can be used to complement the information yielded by the fracture toughness, which depends on several factors, such as the thickness \( B \) of the sample, the shape or size of the notch, etc. [34]. To solve this problem, ASTM E1737-96 [18] establishes a value for the crack length \( a \) (approximately \( 0.5 < a/W < 0.7 \) and \( B = 0.5W \), where \( W \) is the width of the sample) for obtaining the fracture toughness \( K_{IC} \), in order to maintain the small-scale yielding zone.

As shown in Eq. (39), a relationship exists between the specific energy surface \( \gamma_{eff} \) and the minimum considered observation scale \( l_o \). In Fig. 8, it can be observed that the consideration of a minimum size for the fracture \( l_o \) on a grain should mean the effective specific energy of the fracture in this scale \( \gamma_{eff} \). In a similar way, the consideration of a minimum size of fracture in another scale, like one that involves several polycrystalline grains \( l_{o1}, l_{o2}, l_{o3}, \) etc., should take into account the value of an effective specific energy in this other scale, \( \gamma_{eff1}, \gamma_{eff2}, \gamma_{eff3}, \text{etc.} \), in such a way that:

\[
2 \gamma_{eff1}(2 - H) l_{o1}^{H-1} = 2 \gamma_{eff2}(2 - H) l_{o2}^{H-1} = \cdots = \text{const},
\]

although \( l_{o1} \neq l_{o2} \neq l_{o3} \) and \( \gamma_{eff1} \neq \gamma_{eff2} \neq \gamma_{eff3} \). So, the constant does not depend on the single rule of measurement \( l_o \) used in the fractal model, but it depends on the kind of material used in the testing.

Another interpretation of Eq. (38) can be made by splitting the elastic and plastic terms,

\[
J_o = 2 \gamma_e (2 - H) \left(\frac{l_o}{L_o}\right)^{H-1} + \gamma_p (2 - H) \left(\frac{l_o}{L_o}\right)^{H-1},
\]

(41)

**Fig. 8.** Micro structural aspect of the observation scale with different sizes of rules, \( l_o \) to the fractal scaling of the fracture.
From the CFM, one has

\[
J_0 = \frac{K_{oc}^2}{E} + \frac{2A_{pl}}{B_n(w - L_o)}
\]  

Therefore,

\[
J_e = 2\gamma_e(2 - H)\left(\frac{l_o}{ML_0}\right)^{H-1} = \frac{K_{oc}^2}{E}
\]  

and

\[
J_{pl} = \gamma_p(2 - H)\left(\frac{l_o}{ML_0}\right)^{H-1} = \frac{2A_{pl}}{B_n(w - L_o)}
\]

For the particular situation where \(J_0 = J_{IC}\) and \(ML_0 = ML_{oc}\), it can be derived from Eq. (35),

\[
J_{IC} = (2\gamma_e + \gamma_p)(2 - H)\left(\frac{l_o}{ML_{oc}}\right)^{H-1}
\]

and from Eq. (23),

\[
K_{IC} = \sqrt{(2\gamma_e + \gamma_p)E(2 - H)\left(\frac{l_o}{ML_{oc}}\right)^{H-1}}
\]

Therefore, using the fact that once the experimental value of \(J_{IC}\) is determined and the fitting of the \(J-R\) curve has already yielded the values of \(2\gamma_e + \gamma_p, l_o\) and \(H\) for the material, the value \(L_{oc}\) can be calculated as shown in Tables 4 and 5.

5. Discussion of the fractal approaches in fracture mechanics

5.1. The fractal model for the \(J-R\) curve

Fracture mechanics science was originally developed for the study of isotropic situations and homogeneous bodies. It can be studied basically at three scale levels: the micro, meso and macroscopic level.

At the microscopic level, the elastic material is modeled considering Einstein’s solid harmonic approximation where Hooke’s Law is employed for the forces between the chemical bonds of the atoms or molecules [36]. Therefore, the elastic theory is used to make linear approximations and it does not involve micro structural effects of the material.

At the mesoscopic level the equation of energy used for the fracture, (Lamé’s equation, see Ref. [37]), does not take into account effects at the atomic scale involving non-homogeneous situations. Based on the arguments of the last paragraphs, it becomes clear why Herrmnun and coworkers [7,38] and other authors needed to include statistical weights, as a crack growth criterion, for the break of chemical bonds in fracture simulations, as a form of portraying micro structural aspects of the fracture (defects) [37], when using finite difference and finite element methods in computational models.

At the macroscopic level, on the other hand, Griffith’s theory uses a thermodynamic energy balance which is written in a simplified manner:

\[
Xdu = dU + GdA.
\]

It is important to remember that the linear elastic theory of fracture developed by Irwin–Westergaard as the Griffith’s theory, is also a differential theory for the macroscopic scale, which means they are punctual in their local limit. These two approaches involve the micro structural aspects of the fracture, since they take a larger infinitesimal local limit than the linear elastic theory at the atomic scale and the mesoscopic scale. This infinitesimal macroscopic scale is big enough to include \(10^{15}\) particles as the lower thermodynamic limit, where the physical quantity, Fracture Resistance (\(J-R\) curve) portrays aspects of the interaction of the crack with the microstructure of the material.

In this paper, classical fracture mechanics was modified directly using fractal theory, without taking into account more basic formulations, such as the interaction force among particles, or Lamé’s energy equation in the mesoscopic scale as a form to include the ruggedness in the fracture processes.

The idea of relating the morphology (ruggedness) of the fracture surfaces with the physical properties of the materials is not new and has been presented by several authors [9–13,16–41]. The use of the fractality in the fracture surface to quantify the physical process of energy dissipation was approached with two different proposals. The first one was given by Mu and Lung [32] and Lung and Mu [33], who proposed a phenomenological exponential relation between the crack length and the elastic energy release rate in the following form:

\[
G_{IC} = G_{0e}\varepsilon^{1-D}
\]
where \( e \) is the length of the measurement rule. The second proposal was given by Mecholsky et al. [42] and Mandelbrot et al. [39], who suggested an empirical relation between the fractional part of the fractal dimension, \( D^* \), and the fracture toughness \( K_{IC} \) given by,

\[
K_{IC} \sim A(D^*)^{1/2}
\]

(49)

where \( A = E\sqrt{L_0}A \) is a constant, \( E \) the stiffness module, and \( L_0 \) is a parameter that has the unit of length (an atomic characteristic length). The elastic energy release rate is then given by,

\[
G_0 = E\epsilon^2 D^*
\]

(50)

where \( G_0 = K_{IC}^2/E \).

The authors cited above used the Slit Islands Method in their measurements of the fractal dimension \( D \) and it is important to emphasize that both proposals have plausible arguments, in spite of their mathematical discrepancies. Observe that in the proposal of Mu and Lung [32] and Lung and Mu [33] the fractal dimension appears in the exponent of the scale factor, while in the proposals of Mandelbrot et al. [39] and Mecholsky et al. [42] the fractal dimension appears as a multiplying term of the scale factor.

The mathematical expression proposed in this work (Eq. (35)), for the case \( J_0 \equiv G_0 \), is compatible with the two proposals. Therefore, this work can be seen as a unification of these two different approaches in a single mathematical expression. In other words, the two previous proposals are complementary visions of the problem according to the expression deduced in this paper.

A careful experimental interpretation must be done from results obtained in a J–R curve test. The authors mentioned above worked with the concept of \( G \), valid for brittle materials, and not with the concept of \( J \) valid for ductile materials. The experimental results show that for the case of metallic materials the fitting with their expressions are only valid in the initial development of the crack because of the self-similar limit, while self-affinity is a general characteristic of the whole fracture process [35].

5.2. J–R curve tests for metals, weld metallic, polymeric and others materials

The plane strain is a mathematical condition that allows defining a physical quantity called \( K_{IC} \) which does not depend on the thickness of the material. The measure of an average crack size along the thickness of the material, according to the norm ASTM E1737-1996 [18], is taken as an average of the crack size at a certain number of profiles along the thickness. In this way, any profile statistically self-affine, among all the possible profiles that can be obtained in a fracture surface, are statistically equivalent to each other, and give a representative average for the Hurst exponent.

In Fig. 1, it is observed that the crack height (corresponding to CTOD) follows a power law with the scale, \( \varepsilon_h = \varepsilon_v = \varepsilon = l_0/\Delta L_0 \), and can be written as:

\[
\frac{\Delta h}{h_0} = \left( \frac{\Delta L_0}{L_0} \right)^{1-H}
\]

(51)

This relation shows that, while the measurement of the number of units of the crack length, \( N_h = \Delta L_0/l_0 \), in the growth direction grows linearly, the number of units of the crack height, \( N_v = \Delta h/l_0 \), grows with the power of \( 1-H \). If it is considered that, the inverse of the number of crack increments in the growth direction, \( N^{-1}_h = l_0/\Delta L_0 \), is also a measure of strain of the material as the crack grows, and considering that the number of increments of the crack height can be a measure of the amount of pilling up dislocations, in agreement with Eq. (51), then the normal stress is of the kind [43,44]:

\[
\sigma \sim e^{-H}
\]

(52)

Observe that this relation shows a homogeneity in the scale of deformations, similar to the power law hardening equation [30,45]. This shows that the fractal scaling of a rugged fracture surface is related to the power law of the hardening. It is possible that the fractality of the rugged fracture surface is a result of the accumulation of the pilled up dislocations in the hardening of the material before the crack growth.

6. Conclusions

Fracture processes are very complex and not well understood. They involve several factors such as non-homogeneities, microcracks, grain boundaries, emission of sound and radiation, and impurities. The construction of a model that takes into account all the contributing factors is, if not impossible, overwhelming. However, since all of these factors contribute to some extent to the process, it makes sense to presume that their influence leaves traces on the morphology of the fractured surface. This hypothesis is corroborated by the adjustment obtained for the J–R curve shown in Figs. 5 and 6 when the fractality of the surface is taken into account in the fracture mechanics theory.
(i) The model proposed in this work was based on the idea of Mandelbrot [46] and on experimental verification [39] that cracks or fracture surfaces are fractals and that their ruggedness could be modeled analytically in order to be inserted into the equations of fracture mechanics.

(ii) The viability of introducing fractal theory into fracture mechanics is justified facing the countless experimental results of characterization that confirm the fractal nature presented by cracks and fracture surfaces [11,13,41];

(iii) The theory presented in this paper introduces fractal geometry (to describe ruggedness) in the formalism of classical EPFM. The resulting model is consistent with the experimental results, showing that fractal geometry has much to contribute to the advance of this particular science.

(iv) With the model proposed in this paper and using a technique of fractal analysis of ruggedness of the tested samples, it was possible to reproduce the experimental $J-R$ curve of HSLA steels and of polymeric materials. The parameters $l_0$, $H$, and $2\gamma_{\text{eff}} = 2\gamma_c + \gamma_p$, which best adjusted to the curves were obtained by means of a non-linear fitting curve method.

(v) It is shown that the rising of the $J-R$ curve is due to the non-linearity in Griffith–Irwin–Orowan’s energy balance when ruggedness is taken into account.

(vi) The idea of connecting the morphology of a fracture with physical properties of the materials has been done by several authors [9–13,32,41]. This connection is shown in this paper with mathematical rigor.

(vii) Eq. (30) shows that the local ruggedness $dL/dl_0$ depends on the characteristics of the test and the fracture surface. This equation is in opposition to the previous hypothesis that there would be a universal value for the fracture ruggedness dimension for each material.

(viii) It can be easily seen that Eq. (39) is related to the fracture testing along with its effect on the microstructure of the material. This fact allows a very good estimation of the $J-R$ curve. It is simply obtained by measuring the effective energy $\gamma_{\text{eff}}$ on the projected size of the crack $l_0$ and the value of the Hurst exponent, starting from a metallographic analysis of the material, without necessarily conducting a conventional fracture test.

(ix) The technical norms ASTM E813–89 and ASTM E1737–96 suggest an exponential fitting of the type:

$$J_o = C_1 \Delta \epsilon_{\text{eff}}^2$$  (53)

for the $J-R$ curves. They do not supply any explanation for the nature of the coefficients for this fitting. However, by comparing Eq. (53) with Eq. (38), it can be concluded that $C_1 = 2\gamma_{\text{eff}} (2 - H) l_0^{-1}$ and $C_2 = 1 - H$, which explains the physical nature of this parameters;

(x) Therefore, it is important to emphasize that the model proposed in this work illuminates the nature of the coefficients for the adjustment proposed by the fractal model, which is the true influence of ruggedness in the rising of the $J-R$ curve. The application of this model in the practice of fracture testing can be used in future, since the techniques for obtaining the experimental parameters, $l_0$, $H$, and $\gamma_{\text{eff}}$ can be accomplished with the necessary accuracy;

(xi) The method for obtaining the $J-R$ curves proposed in this paper does not intend to substitute the current experimental method used in fracture mechanics, as presented by the ASTM norms. However, it can give a greater margin of confidence in experimental results, and also when working with the microstructure of the materials. For instance, in search of new materials with higher fracture toughness, once the model explains micro and macroscopically the behavior of $J-R$ curves.

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